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On Cosmic No-hair in Bimetric Gravity and the Higuchi Bound

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Abstract

We study the cosmic no-hair in the presence of spin-2 matter, i.e. in bimetric gravity. We obtain stable de Sitter solutions with the cosmological constant in the physical sector and find an evidence that the cosmic no-hair is correct. In the presence of the other cosmological constant, there are two branches of de Sitter solutions. Under anisotropic perturbations, one of them is always stable and there is no violation of the cosmic no-hair at the linear level. The stability of the other branch depends on parameters and the cosmic no-hair can be violated in general. Remarkably, the bifurcation point of two branches exactly coincides with the Higuchi bound. It turns out that there exists a de Sitter solution for which the cosmic no-hair holds at the linear level and the effective mass for the anisotropic perturbations is above the Higuchi bound.

I. INTRODUCTION

It is well recognized that the large scale structure of the universe stems from primordial fluctuations generated quantum mechanically during inflation. Remarkably, the nature of primordial fluctuations is independent of initial conditions. This nice feature can be associated with the conjecture that the initial anisotropy and inhomogeneity rapidly disappear. This is called the cosmic no-hair conjecture. The cosmic no-hair is proved in an ideal situation [1]. Namely, a homogeneous expanding spacetime with a cosmological constant rapidly approaches de Sitter spacetime, i.e., the initial anisotropy decays in a Hubble time, when we assume that matter satisfies the strong and dominant energy conditions. In general, however, it is not clear whether the cosmic no-hair conjecture is correct or not. In fact, a counter example to this conjecture was found [2]. There, spin-1 gauge fields remain during inflation and the anisotropy does not necessarily vanish. Moreover, it turned out that anti-symmetric tensor fields can also generate the anisotropy [3]. Hence, it is natural to explore the possibility that a symmetric spin-2 tensor as matter causes the violation of the cosmic no-hair conjecture. Historically, a model of massive spin-2 matter has been proposed as that of meson [4], which can be regarded as bimetric gravity consisting of the physical metric and the other spin-2 tensor field. In order to treat the spin-2 matter, therefore, we need to construct a consistent ghost-free theory of bimetric gravity. Fortunately, this task has been accomplished recently [5–9].

Given a consistent model of spin-2 matter, we can study the cosmic no-hair conjecture. There are some reasons that we expect the conjecture can be violated. In the presence of spin-2 matter, it is inevitable that gravitons have the mass as the consequence of mixing between the physical metric and the other spin-2 tensor field. When we consider massive gravitons in an expanding spacetime, the decay time scale of the anisotropy is determined by comparing Hubble scale with the effective mass of gravitons. For example, by taking the couplings of the physical metric and the spin-2 matter small, the Hubble friction term might be dominant compared with the effective mass term in the equation of motion then the decay time scale becomes much longer than Hubble time scale. Besides the above one, there may be the violation of the energy conditions in the presence of the spin-2 matter [10]. Since the energy conditions are assumed in the proof of [1], it is not apparent whether the cosmic no-hair holds or not in bimetric gravity.

In this paper, we consider a cosmological constant in bimetric gravity as the limit of slow roll inflation. First, we concretely reveal the property of de Sitter solutions in bimetric gravity. Then, we investigate the fate of the anisotropy perturbatively. We stress that it is important to study the background geometry in detail because the effective mass of gravitons can depend on the background geometry. Since, in known cases, the violation of the cosmic no-hair appears already at the linear level, we expect the linear analysis reflects the feature at the nonlinear level.

When we consider massive gravitons in de Sitter spacetime, we also need to care about the fact that the helicity-0 mode of massive gravitons becomes a ghost when the effective mass is below the Higuchi bound [11–14]. Note that this ghost is different from a Boulware-Deser type ghost [15] which is already removed by construction. Since there is no a priori reason to forbid the mass of gravitons violating the Higuchi bound, we also check if the effective mass satisfies the Higuchi bound.

We organize the paper as follows. In section II, we present ghost-free bimetric gravity and derive basic equations needed for the analysis. In section III, we study the cosmic no-hair in bimetric gravity in the presence of the cosmological constant in the physical sector. We find that de Sitter solutions are stable and the small anisotropy rapidly decays. In section IV, we introduce the other cosmological constant and investigate the stability of de Sitter solutions and the fate of the anisotropy. We also study whether the Higuchi bound is satisfied or not. The final section is devoted to the conclusion. In appendix A, we derive a set of equations used in the text.

II. BIMETRIC GRAVITY

In this section, we introduce bimetric gravity [8, 9] as a model of spin-2 matter and provide basic formulae. Historically, after the pioneering work [4], bimetric gravity has been studied from time to time [16, 17]. The model can be generalized to that of ghost-free multi-spin-2 matter [18, 19].

Let us represent the physical metric and the other metric as $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively. Note that we regard the other metric $f_{\mu\nu}$ as the spin-2 matter. We consider bimetric gravity

with cosmological constants

$$\begin{aligned} S &= \frac{M_g^2}{2}\int d^4x\sqrt{-g}(R[g_{\mu\nu}]-2\Lambda_g)+\frac{M_f^2}{2}\int d^4x\sqrt{-f}(R[f_{\mu\nu}]-2\Lambda_f) \\ &\quad +m^2M_e^2\int d^4x\sqrt{-g}\sum_{n=1}^{3}\alpha_nF_n[L^\mu_\nu]\ , \end{aligned} \tag{1}$$

where M_g and M_f are Planck constants of $g_{\mu\nu}$ and $f_{\mu\nu}$, and R is the scalar curvature constructed from each metric. The interaction terms of the metrics are defined as

$$F_n[X^\mu_\nu]=\frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)X_{\mu_1}^{\mu_{\sigma(1)}}X_{\mu_2}^{\mu_{\sigma(2)}}\cdots X_{\mu_n}^{\mu_{\sigma(n)}}\ ,$$

$$L^\mu_\nu=\delta^\mu_\nu-(\sqrt{g^{-1}f})^\mu_\nu\ .$$

This combination of interaction terms gives no Boulware-Deser ghost [9]. Here, m^2 is a coupling constant of the metrics and $\{\alpha_n\}_{n=1,2,3}$ are arbitrary constants. We define the reduced Planck constant M_e as

$$\frac{1}{M_e^2}=\frac{1}{M_g^2}+\frac{1}{M_f^2}\ ,$$

where M_e is chosen so that m coincides with the Fierz-Pauli mass [20] when we take the massive gravity limit. Note that we can regard Λ_g as the potential energy of a scalar field in the slow roll approximation coupled to the physical metric $g_{\mu\nu}$ as in general relativity.

In this paper, we consider the simplest case $\alpha_2=1$, $\alpha_1=\alpha_3=0$. Then, the action is written as

$$\begin{aligned} S &= \frac{M_g^2}{2}\int d^4x\sqrt{-g}(R[g_{\mu\nu}]-2\Lambda_g)+\frac{M_f^2}{2}\int d^4x\sqrt{-f}(R[f_{\mu\nu}]-2\Lambda_f) \\ &\quad +m^2M_e^2\int d^4x\sqrt{-g}F_2[L^\mu_\nu]\ , \end{aligned} \tag{2}$$

where

$$F_2[L^\mu_\nu]=\frac{1}{2}([L]^2-[L^2])\ ,\qquad [L]=L^\mu_\mu\ ,\qquad [L^2]=L^\mu_\nu L^\nu_\mu\ .$$

We now present basic equations and derive formulae which will be used in the later analysis.

A. de Sitter solutions in bimetric gravity

In this subsection, we consider homogeneous and isotropic solutions in bimetric gravity [10, 21–25]. We derive equations of motion and show the solutions are de Sitter spacetimes.

We take the homogeneous and isotropic metric ansatz for $g_{\mu\nu}$ and $f_{\mu\nu}$,

$$ds^2 = -N^2(t)dt^2 + e^{2\alpha(t)}[dx^2 + dy^2 + dz^2] \ , \tag{3}$$

and

$$ds'^2 = -M^2(t)dt^2 + e^{2\beta(t)}[dx^2 + dy^2 + dz^2] \ , \tag{4}$$

respectively. M, N are lapse functions and α, β describe the isotropic expansion of each metric. Substituting the metric ansatz into the action, we obtain the Lagrangian

$$\begin{aligned}\mathcal{L} \ = \ & M_g^2 e^{3\alpha}\Big[-\frac{3\dot{\alpha}^2}{N} - N\Lambda_g\Big] + M_f^2 e^{3\beta}\Big[-\frac{3\dot{\beta}^2}{M} - M\Lambda_f\Big] \\ & + m^2 M_e^2 N e^{3\alpha}\big[6 - 9\epsilon + 3\epsilon^2 + \gamma(-3+3\epsilon)\big] \ ,\end{aligned} \tag{5}$$

where

$$\gamma = \frac{M}{N} \ , \qquad \epsilon = e^{\beta-\alpha} \ . \tag{6}$$

Taking the variation with respect to each variable, we obtain the equations of motion for α and β

$$\Big(\frac{\alpha'}{N}\Big)' - \xi a_g (M - N\epsilon)\Big(\frac{3}{2} - \epsilon\Big) = 0 \ , \tag{7}$$

$$\Big(\frac{\beta'}{M}\Big)' + \xi(1-a_g)\epsilon^{-3}(M - N\epsilon)\Big(\frac{3}{2} - \epsilon\Big) = 0 \ , \tag{8}$$

and two constraints

$$\Big(\frac{\alpha'}{N}\Big)^2 = \lambda_g + \xi a_g (2-\epsilon)(\epsilon - 1) \ , \tag{9}$$

$$\Big(\frac{\beta'}{M}\Big)^2 = \lambda_f + \xi(1-a_g)\epsilon^{-3}(1-\epsilon) \ , \tag{10}$$

where we normalized parameters and time with M_e as follows:

$$a_g = \frac{M_e^2}{M_g^2} \ , \qquad \xi = \frac{m^2}{M_e^2} \ , \qquad \lambda_g = \frac{\Lambda_g}{3M_e^2} \ , \qquad \lambda_f = \frac{\Lambda_f}{3M_e^2} \ , \qquad {}' = \frac{1}{M_e}\frac{\mathrm{d}}{\mathrm{d}t} \ . \tag{11}$$

We notice that a_g can take the value in the range $0 < a_g < 1$ from the definition of M_e. The detailed derivation can be found in Appendix A.

In bimetric gravity, the diagonal part of general coordinate invariance is preserved. Hence, the two constraints contain a first class constraint and a second class constraint. Thus, there exists a secondary constraint. Now, from (7) and (9) (or (8) and (10)), we can deduce the equation

$$\xi\Big(\frac{3}{2} - \epsilon\Big)\Big(\frac{\beta' e^{\beta}}{M} - \frac{\alpha' e^{\alpha}}{N}\Big) = 0 \ . \tag{12}$$

The first factor can be taken to be zero. However, this is a special solution and it is known that this leads to a pathology [26–30]. Hence, we take the following branch

$$M = \frac{\beta'}{\alpha'} N\epsilon \ . \tag{13}$$

This is nothing but the condition determining the Lagrange multiplier. From (9), (10) and (13), we obtain the secondary constraint

$$g(\epsilon) = (\lambda_f + \xi a_g)\epsilon^3 - 3\xi a_g \epsilon^2 + [-\lambda_g + 2\xi a_g - \xi(1-a_g)]\epsilon + \xi(1-a_g) = 0 \ . \tag{14}$$

From the definition of ϵ, ϵ should be positive and hence we should look for the positive roots of the algebraic equation $g(\epsilon) = 0$. Since ξ, a_g, λ_g and λ_f are constants, a positive root of $g(\epsilon) = 0$ is also a constant which we represent ϵ_0. Then, taking the derivative of the definition of ϵ, we derive $\alpha' = \beta'$ and hence $M = N\epsilon_0$. Now, we take a gauge $N = 1$ using the gauge degree of freedom. Then, we get $M = \epsilon_0 =$ constant. From (7) and (8), we can deduce $\alpha'' = \beta'' = 0$ which can be solved as $\alpha = H_0 M_e t$, $\beta = H_0 M_e t + \log(\epsilon_0)$, where H_0 is Hubble scale which is determined from the constraints as

$$\begin{aligned} H_0^2 &= \lambda_g + \xi a_g (2-\epsilon_0)(\epsilon_0 - 1) \\ &= \lambda_f \epsilon_0^2 + \xi(1-a_g)\frac{1-\epsilon_0}{\epsilon_0} \ . \end{aligned} \tag{15}$$

Thus, we obtained two de Sitter spacetimes with the relation $f_{\mu\nu} = \epsilon_0^2 \, g_{\mu\nu}$ provided that ϵ_0 is a positive root of $g(\epsilon) = 0$ and $H_0^2 > 0$ holds for ϵ_0.

B. Fate of the anisotropy

In this subsection, we consider the anisotropy perturbatively and examine how the anisotropy evolves. We also derive the effective mass of the massive graviton.

We take the anisotropic metric ansatz

$$ds^2 = -N^2(t)dt^2 + e^{2\alpha(t)}[e^{-4\sigma(t)}dx^2 + e^{2\sigma(t)}(dy^2 + dz^2)] \ , \tag{16}$$

and

$$ds'^2 = -M^2(t)dt^2 + e^{2\beta(t)}[e^{-4\lambda(t)}dx^2 + e^{2\lambda(t)}(dy^2 + dz^2)] \ , \tag{17}$$

where σ and λ describes the anisotropic expansion of each metric. Here we assume the anisotropy is small. Substituting the metric ansatz into the action and dropping the higher

order terms, we can derive the quadratic Lagrangian

$$\delta^2\mathcal{L} \;=\; M_g^2 e^{3\alpha}\frac{3\dot{\sigma}^2}{N} + M_f^2 e^{3\beta}\frac{3\dot{\lambda}^2}{M} + m^2 M_e^2 N e^{3\alpha}\left[-9\epsilon + 3\epsilon^2 + 3\gamma\epsilon\right]q^2 \ , \tag{18}$$

where we defined the new variable

$$q = \lambda - \sigma \ . \tag{19}$$

Note that σ and λ can be regarded as zero modes of gravitons. From the above action, we can deduce the equations for σ as

$$\sigma'' + 3H_0\sigma' - \xi a_g \epsilon_0(3-2\epsilon_0)q = 0 \ , \tag{20}$$

and for λ as

$$\lambda'' + 3H_0\lambda' + \xi(1-a_g)\frac{1}{\epsilon_0}(3-2\epsilon_0)q = 0 \ . \tag{21}$$

By taking the difference of (21) and (20), it is easy to obtain

$$q'' + 3H_0 q' + \xi\left[a_g\epsilon_0 + (1-a_g)\frac{1}{\epsilon_0}\right](3-2\epsilon_0)q = 0 \ . \tag{22}$$

From this equation, we can read off the effective mass of the massive graviton as

$$m_{\text{eff}}^2 = \xi\left[a_g\epsilon_0 + (1-a_g)\frac{1}{\epsilon_0}\right](3-2\epsilon_0) \ . \tag{23}$$

Since the effective mass is different from the bare mass ξ, it is non-trivial if the effective mass is less than Hubble scale even if the bare mass is so. By making the combination $(20)\times 1/a_g + (21)\times \epsilon_0^2/(1-a_g)$, we have

$$\left[e^{3H_0 t}\left(\frac{\sigma'}{a_g} + \epsilon_0^2\frac{\lambda'}{1-a_g}\right)\right]' = 0 \ . \tag{24}$$

This leads to a conserved quantity

$$E = e^{3H_0 t}\left(\frac{\sigma'}{a_g} + \epsilon_0^2\frac{\lambda'}{1-a_g}\right) \tag{25}$$

which means the mode

$$\frac{\sigma'}{a_g} + \epsilon_0^2\frac{\lambda'}{1-a_g} \tag{26}$$

corresponds to the massless graviton. The existence of the massless mode is a reflection of the diagonal general coordinate invariance. From the conservation law (25), we see that this mode vanishes exponentially fast.

If we substitute $q = e^{i\omega t}$ into eq. (22), we obtain

$$q = A \exp i\omega_+ t + B \exp i\omega_- t \ , \tag{27}$$

where

$$\omega_\pm = i\frac{3H_0}{2} \pm \sqrt{m_{\rm eff}^2 - \frac{9H_0^2}{4}} \tag{28}$$

and A, B are integral constants. If $m_{\rm eff}^2$ is negative, q exponentially grows like

$$q \sim B \exp\left(\sqrt{|m_{\rm eff}^2| + \frac{9H_0^2}{4}} - \frac{3H_0}{2}\right)t \ . \tag{29}$$

Inversely, if $m_{\rm eff}^2$ is positive, q exponentially decays. When $m_{\rm eff}^2 - \frac{9H_0^2}{4} > 0$, the decay time scale τ is $\tau = 2/3H_0$. On the other hand, if $m_{\rm eff}^2 - \frac{9H_0^2}{4} < 0$, the time scale is evaluated as

$$\tau^{-1} = |\omega_-| = \frac{3H_0}{2} - \sqrt{\frac{H_0^2}{4} + (2H_0^2 - m_{\rm eff}^2)} \ . \tag{30}$$

Therefore, the decay time scale of the anisotropy τ is shorter than Hubble time scale $1/H_0$ for $m_{\rm eff}^2 > 2H_0^2$ and the opposite holds for $m_{\rm eff}^2 < 2H_0^2$.

III. DECAY OF THE ANISOTROPY: CASES $\lambda_f = 0$

First, we consider the situation $\lambda_f = 0$. The constant λ_g can be regarded as the potential energy of a scalar field coupled to $g_{\mu\nu}$ in the slow roll approximation. We prove that there exist a de Sitter solution for $\lambda_g > 0$ and the solution is stable under the anisotropic perturbations. We also see that the effective mass of the massive graviton is bounded from below $m_{\rm eff}^2 > 3H_0^2$. This suggests that the anisotropy rapidly decays in a Hubble time.

When we take $\lambda_f = 0$, (14) and (15) become

$$g(\epsilon) = \xi a_g \epsilon^3 - 3\xi a_g \epsilon^2 + [-\lambda_g + 2\xi a_g - \xi(1-a_g)]\epsilon + \xi(1-a_g) = 0 \tag{31}$$

and

$$\begin{aligned} H_0^2 &= \lambda_g + \xi a_g (2-\epsilon_0)(\epsilon_0 - 1) \\ &= \xi(1-a_g)\frac{1-\epsilon_0}{\epsilon_0} \ . \end{aligned} \tag{32}$$

From the second line of (32), we see that ϵ_0 should be less than 1 so that H_0 is a real number. Then, from the first line of (32), λ_g should have a positive lower bound. We assume that λ_g is positive in the following. Then, we obtain

$$g(0) = \xi(1-a_g) > 0 \ , \qquad g(1) = -\lambda_g < 0 \ , \qquad g(\epsilon) \to +\infty \ \text{ as } \ \epsilon \to +\infty \ .$$

Thus, there are a positive root smaller than 1 and a positive root larger than 1. The root larger than 1 does not satisfy the condition $H_0^2 > 0$. It turned out that there is a single positive root ϵ_0 in the range $0 < \epsilon_0 < 1$ where H_0 is a real number in the case λ_g is positive.

Next, we will see that the de Sitter solution derived above is always stable under the anisotropic perturbations. Apparently, massless modes rapidly decay in Hubble time scale. Then the stability under the perturbations of σ and λ is determined by the sign of the mass term of the perturbation equation (22) as mentioned in Sec. II B or the sign of

$$m_{\rm eff}^2 = \xi\Big[a_g\epsilon_0 + (1-a_g)\frac{1}{\epsilon_0}\Big](3-2\epsilon_0)\ .$$

Since the de Sitter solution satisfies $0 < \epsilon_0 < 1$, $m_{\rm eff}^2$ is positive. Therefore, the de Sitter solution is stable under the perturbations of σ and λ.

Furthermore, we can prove that $m_{\rm eff}^2$ is bounded from below $m_{\rm eff}^2 > 3H_0^2$. To show this, let us define

$$h(\epsilon_0) = \frac{m_{\rm eff}^2}{H_0^2} = \frac{[a_g\epsilon_0^2 + (1-a_g)](3-2\epsilon_0)}{(1-a_g)(1-\epsilon_0)}\ , \tag{33}$$

where we used the second line of (32). It is straightforward to calculate the derivative of $h(\epsilon_0)$,

$$\frac{\rm d}{{\rm d}\epsilon_0}h(\epsilon_0) = \frac{4a_g\epsilon_0(\epsilon_0-\frac{9}{8})^2 + a_g\epsilon_0\frac{15}{16} + (1-a_g)}{(1-a_g)(1-\epsilon_0)^2}\ . \tag{34}$$

Since this is manifestly positive in the range $0 < \epsilon_0 < 1$, we have the inequality $h(\epsilon_0) > h(0) = 3$, that is,

$$m_{\rm eff}^2 > 3H_0^2\ . \tag{35}$$

The effective mass of the massive graviton is bounded by Hubble scale from below. Using the analysis in Sec. II B, we can see that the anisotropy rapidly decays in a Hubble time.

IV. DECAY OF THE ANISOTROPY: CASES $\lambda_f \neq 0$

In this section, we construct de Sitter solutions with $\lambda_f \neq 0$. Then, we check the perturbative stability of the de Sitter solutions. Finally, we examine if the effective mass of the massive graviton can be smaller than Hubble scale.

A. de Sitter solutions

We study de Sitter solutions and give a classification of them. What we should check is whether roots of $g(\epsilon)=0$ are positive and satisfy $H_0^2>0$.

1. *When are roots of $g(\epsilon)=0$ positive?*

Since the behavior of $g(\epsilon)$ is largely determined by the leading term, $\lambda_f+\xi a_g$, we discuss the following three cases separately.

1. In the case $\lambda_f>-\xi a_g$, the coefficient of the leading term in $g(\epsilon)$ is positive, which indicates
$$g(\epsilon)\to-\infty \ \text{ as } \ \epsilon\to-\infty\ , \qquad g(\epsilon)\to+\infty \ \text{ as } \ \epsilon\to+\infty\ .$$
Combining the above with $g(0)=\xi(1-a_g)>0$, we see that there always exists a negative root. Since $g''(0)=-6\xi a_g<0$, the inflection point must exist in the positive side of ϵ. Therefore, the number of positive solutions can be characterized by the discriminant of $g(\epsilon)=0$. If the discriminant is zero, a multiple positive root exists. On the other hand, if the discriminant is positive, two positive roots exist. The discriminant of $g(\epsilon)=0$ is given by
$$\begin{aligned} D \ =\ & -27(1-a_g)^2\Big(\frac{\lambda_f}{\xi}+a_g\Big)^2 \\ & +2\bar{c}[2\bar{c}^2+27a_g(1-a_g)]\Big(\frac{\lambda_f}{\xi}+a_g\Big)+9a_g^2[\bar{c}^2+12a_g(1-a_g)]\ , \end{aligned} \tag{36}$$
where we defined
$$\bar{c}=\lambda_g/\xi-2a_g+(1-a_g)\ . \tag{37}$$
The condition that the discriminant is non negative reads
$$\lambda_-\le\lambda_f\le\lambda_+\ , \tag{38}$$
where we defined
$$\frac{\lambda_\pm}{\xi}+a_g=\frac{1}{27(1-a_g)^2}\big\{\bar{c}[2\bar{c}^2+27a_g(1-a_g)]\pm2[\bar{c}^2+9a_g(1-a_g)]^{\frac{3}{2}}\big\}\ . \tag{39}$$
We can see $\lambda_-<-\xi a_g$ and $\lambda_+>-\xi a_g$ from (39) taking into account the inequality
$$\big|2[\bar{c}^2+9a_g(1-a_g)]^{\frac{3}{2}}\big|-\big|\bar{c}[2\bar{c}^2+27a_g(1-a_g)]\big|>0\ .$$

Thus, for $\lambda_f = \lambda_+$, there exists a single multiple positive root of $g(\epsilon) = 0$. Since we are considering the range $\lambda_f > -\xi a_g$, there exist two positive roots for $-\xi a_g < \lambda_f < \lambda_+$.

2. In the case $\lambda_f = -\xi a_g$, $g(\epsilon)$ becomes the quadratic function of ϵ. Since the coefficient of the leading term $-3\xi a_g$ is negative and $g(0) = \xi(1 - a_g) > 0$, there exists a single positive root.

3. In the case $\lambda_f < -\xi a_g$, the coefficient of the leading term in $g(\epsilon)$ is negative, which leads to

$$g(\epsilon) \to +\infty \ \text{ as } \ \epsilon \to -\infty \ , \qquad g(\epsilon) \to -\infty \ \text{ as } \ \epsilon \to +\infty \ .$$

Because of the fact $g(0) = \xi(1 - a_g) > 0$, there always exists a positive root. Since $g''(0) = -6\xi a_g < 0$, the inflection point exists on the negative side of ϵ in this case. Thus, other possible roots should be negative. Namely, there exists a single positive root for $\lambda_f < -\xi a_g$.

We found that two positive roots exist for $-\xi a_g < \lambda_f < \lambda_+$ and a single positive root exists for $\lambda_f \leq -\xi a_g$ and $\lambda_f = \lambda_+$.

Next, we check whether these roots satisfy the condition $H_0^2 > 0$.

2. Is $H_0^2 > 0$ satisfied ?

Rewriting the first line of (15) as

$$\begin{aligned} H_0^2 &= \lambda_g + \xi a_g (2 - \epsilon_0)(\epsilon_0 - 1) \\ &= \xi a_g \left[-\left(\epsilon_0 - \frac{3}{2} \right)^2 + \frac{\lambda_g}{\xi a_g} + \frac{1}{4} \right] \ , \end{aligned} \tag{40}$$

we see that $\lambda_g > -\xi a_g/4$ is at least needed for $H_0^2 > 0$. Therefore we assume $\lambda_g > -\xi a_g/4$ below. Then, we can factorize (40) as

$$H_0^2 = -\xi a_g (\epsilon_0 - \epsilon_p)(\epsilon_0 - \epsilon_m) \ , \tag{41}$$

where we defined

$$\epsilon_p = \frac{3}{2} + \sqrt{\frac{\lambda_g}{\xi a_g} + \frac{1}{4}} \ , \quad \epsilon_m = \frac{3}{2} - \sqrt{\frac{\lambda_g}{\xi a_g} + \frac{1}{4}} \ . \tag{42}$$

Note that ϵ_p and ϵ_m do not depend on λ_f. Thus, in order to have $H_0^2 > 0$, we have to seek positive roots of $g(\epsilon) = 0$ in the range

$$\epsilon_m < \epsilon_0 < \epsilon_p \ . \tag{43}$$

As we discussed in the previous subsection, $\lambda_f \le \lambda_+$ is needed for the existence of positive roots. We first consider the case $\lambda_f = \lambda_+$ for which there exists a single positive root. In this case, we have to solve $g(\epsilon_*) = g'(\epsilon_*) = 0$ which give rise to the equation

$$a_g \epsilon_*^2 + \frac{2}{3}\bar{c}\epsilon_* - (1-a_g) = 0 \ . \tag{44}$$

The positive root of this equation is given by

$$\epsilon_* = \frac{-\bar{c} + \sqrt{\bar{c}^2 + 9a_g(1-a_g)}}{3a_g} \ . \tag{45}$$

Thus, we see

$$H_0^2(\epsilon_*) = \frac{2\xi(\frac{\lambda_g}{\xi} + \frac{a_g}{4})\sqrt{\bar{c}^2 + 9a_g(1-a_g)}}{(\frac{\lambda_g}{\xi} + \frac{a_g}{4}) + \frac{9}{4}a_g + (1-a_g) + \sqrt{\bar{c}^2 + 9a_g(1-a_g)}} > 0 \ . \tag{46}$$

Therefore, the inequality $\epsilon_m < \epsilon_* < \epsilon_p$ must hold.

As we decrease λ_f with fixing λ_g, a_g, ξ, the discriminant of $g(\epsilon) = 0$ becomes positive. Thus, there will be two positive roots until λ_f reaches $-\xi a_g$. We shall call smaller one inner root, $\epsilon_{\rm in}$, and the other one outer root, $\epsilon_{\rm out}$. We note that $\epsilon_{\rm in}$ is always smaller than ϵ_* and $\epsilon_{\rm out}$ is always larger than ϵ_* because

$$g(0) = \xi(1-a_g) > 0, \qquad g(\epsilon_*) = \epsilon_*^3(\lambda_f - \lambda_+) < 0, \qquad g(\epsilon) \to +\infty \ \text{ as } \ \epsilon \to +\infty.$$

We can regard $\lambda_f \le -\xi a_g$ case as the inner root because the inner root is continuously connected to the positive root for $\lambda_f < -\xi a_g$ when λ_f crosses $-\xi a_g$ below.

We shall evaluate the first derivative of ϵ_0 with respect to λ_f since we want to know the behavior of the roots when we decrease λ_f. Differentiating $g(\lambda_f, \epsilon_0(\lambda_f)) = 0$ with respect to λ_f

$$\left.\frac{\mathrm{d}g(\lambda_f, x(\lambda_f))}{\mathrm{d}\lambda_f}\right|_{x=\epsilon_0} = 0 \ , \tag{47}$$

we obtain

$$\frac{\mathrm{d}\epsilon_0}{\mathrm{d}\lambda_f} = -\frac{\epsilon_0^3}{\left.\frac{\mathrm{d}g(x)}{\mathrm{d}x}\right|_{x=\epsilon_0}} \ . \tag{48}$$

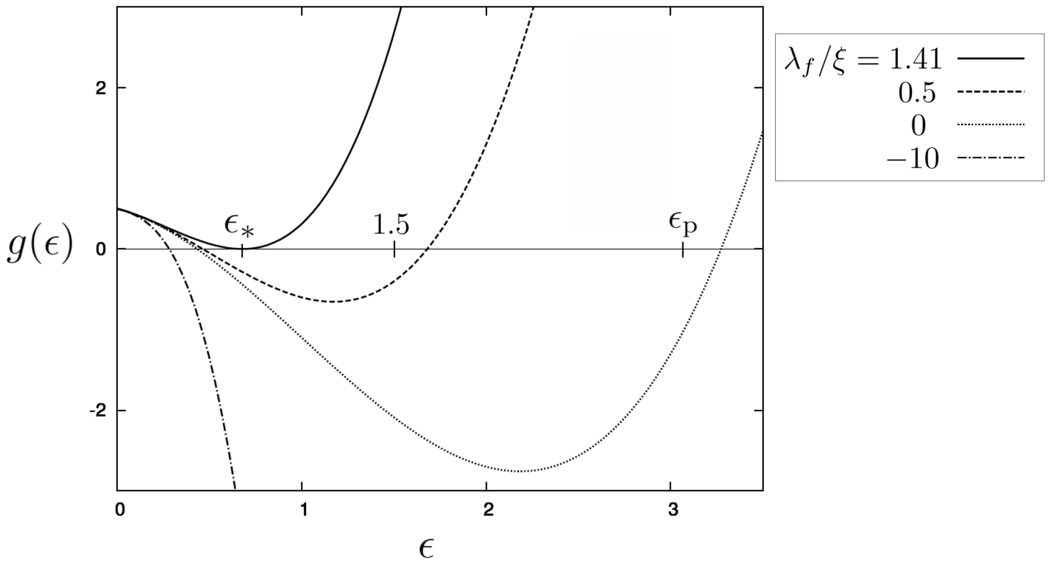

FIG. 1: We plotted $g(\epsilon)$ for $\lambda_g \geq 2\xi a_g$. We set $a_g = 0.5$, $\lambda_g/\xi = 1.1$. Then $\lambda_+/\xi \simeq 1.41$. As λ_f decreases, the outer root increases and the inner root decreases. When λ_f reaches λ_p, the outer root crosses ϵ_p above and $H_0^2(\epsilon_{\rm out})$ becomes negative. But the inner root always satisfies $H_0^2(\epsilon_{\rm in}) > 0$ since ϵ_m is non positive.

First, we discuss the outer root. Since $g(\epsilon_*) < 0$ and $g(\epsilon) \to +\infty$ as $\epsilon \to +\infty$, the outer root always satisfies

$$\left.\frac{{\rm d}g(x)}{{\rm d}x}\right|_{x=\epsilon_{\rm out}} > 0 \ . \tag{49}$$

Then, from (48), we can see

$$\frac{{\rm d}\epsilon_{\rm out}}{{\rm d}\lambda_f} < 0 \ . \tag{50}$$

Therefore, $\epsilon_{\rm out}$ starts from ϵ_* at $\lambda_f = \lambda_+$ and $\epsilon_{\rm out}$ monotonically increases as λ_f decreases. We can expect that $\epsilon_{\rm out}$ sometime reaches ϵ_p. Indeed, $\epsilon_{\rm out}$ reaches ϵ_p when λ_f becomes small as

$$\lambda_p = \xi(1-a_g)\frac{\epsilon_p - 1}{\epsilon_p^3} > 0 \ , \tag{51}$$

where we used the fact $H_0^2 = 0$ at ϵ_p. Therefore, $\epsilon_{\rm out}$ exists in the range (ϵ_m, ϵ_p) if and only if $\lambda_f > \lambda_p$. We mention that $\lambda_p \to +0$ when $\lambda_g \to +\infty$ since $\epsilon_p \to +\infty$ (see (42)).

Next, we discuss the inner root. In turn, since $g(0) = \xi(1-a_g) > 0$ and $g(\epsilon_*) < 0$, the

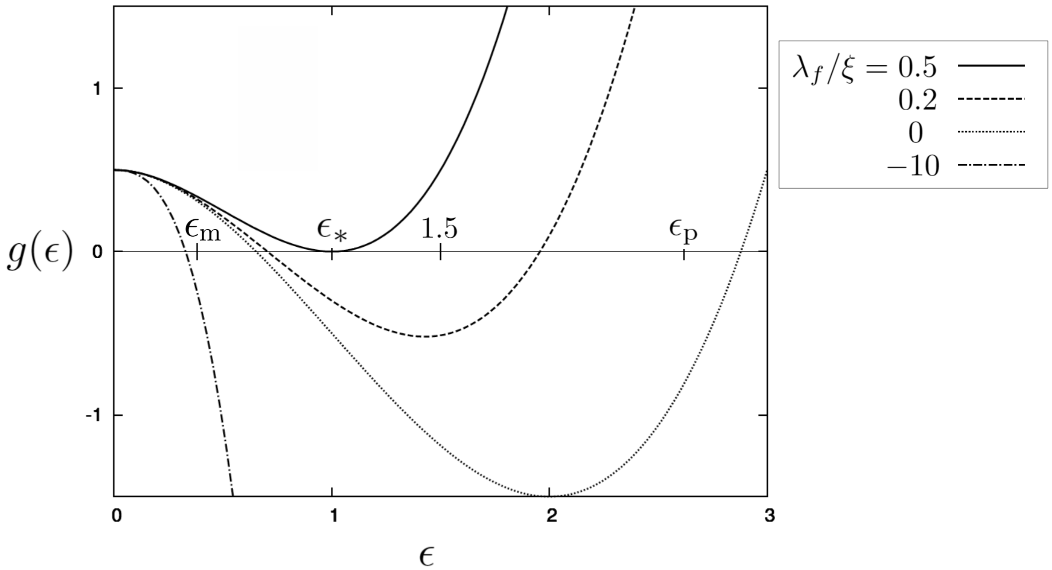

FIG. 2: We plotted $g(\epsilon)$ for $-\frac{1}{4}\xi a_g < \lambda_g < 2\xi a_g$. We set $a_g = 0.5$, $\lambda_g/\xi = 0.5$. Then $\lambda_+/\xi = 0.5$. As λ_f decreases, the outer root increases and the inner root decreases. When λ_f reaches λ_p, the outer root crosses ϵ_p above and $H_0^2(\epsilon_{\rm out})$ becomes negative. When λ_f reaches λ_m, the inner root crosses ϵ_m below and $H_0^2(\epsilon_{\rm in})$ becomes negative.

inner root always satisfies

$$\left.\frac{{\rm d}g(x)}{{\rm d}x}\right|_{x=\epsilon_{\rm in}} < 0 \ . \tag{52}$$

Then from (48), we can see

$$\frac{{\rm d}\epsilon_{\rm in}}{{\rm d}\lambda_f} > 0 \ . \tag{53}$$

Therefore, $\epsilon_{\rm in}$ starts from ϵ_* at $\lambda_f = \lambda_+$ and monotonically decreases as λ_f decreases. Note that $\epsilon_{\rm in} \to \left(\frac{\xi(1-a_g)}{|\lambda_f|}\right)^{1/3} \to +0$ as $\lambda_f \to -\infty$. We can expect that $\epsilon_{\rm in}$ sometime reaches ϵ_m. To see this, we need to notice that

$$\epsilon_m = \frac{3}{2} - \sqrt{\frac{\lambda_g}{\xi a_g} + \frac{1}{4}} = \frac{3}{2} - \sqrt{\frac{\lambda_g - 2\xi a_g}{\xi a_g} + \frac{9}{4}} \tag{54}$$

changes the sign at $\lambda_g = 2\xi a_g$. Hence, we can consider the following two cases.

1. In the case $\lambda_g \geq 2\xi a_g$, ϵ_m is non positive. Then $\epsilon_{\rm in}$ cannot reach ϵ_m when we decrease λ_f. Therefore, $\epsilon_{\rm in}$ always exists in the range (ϵ_m, ϵ_p) and satisfies $H_0^2(\epsilon_{\rm in}) > 0$ (see Fig. 1).

2. In the case $-\frac{1}{4}\xi a_g < \lambda_g < 2\xi a_g$, ϵ_m is positive. Then $\epsilon_{\rm in}$ can reach ϵ_m when we decrease λ_f. Indeed, $\epsilon_{\rm in}$ reaches ϵ_m when λ_f becomes small as

$$\lambda_m = \xi(1-a_g)\frac{\epsilon_m - 1}{\epsilon_m^3} \ , \tag{55}$$

where we used the fact $H_0^2 = 0$ at ϵ_m. Therefore, $\epsilon_{\rm in}$ exists in the range (ϵ_m, ϵ_p) and satisfies $H_0^2 > 0$ if and only if $\lambda_f > \lambda_m$. We mention that $\lambda_m \to -\infty$ when $\lambda_g \to 2\xi a_g - 0$ because $\epsilon_m \to +0$ (see (54)). In Fig. 2, we illustrate these features.

We note that $\lambda_p > \lambda_m$ when $-\frac{1}{4}\xi a_g < \lambda_g < 2\xi a_g$. We can see this from the definitions of λ_p and λ_m as

$$\lambda_p - \lambda_m = \xi(1-a_g)\frac{8\left(\frac{\lambda_g}{\xi a_g} + \frac{1}{4}\right)^{\frac{3}{2}}}{\left(2 - \frac{\lambda_g}{\xi a_g}\right)^3} > 0 \ . \tag{56}$$

We summarize the results derived in this subsection in Table I, Table II and Fig. 3.

TABLE I: For $\lambda_g \geq 2\xi a_g$

	inner	outer
$\lambda_+ < \lambda_f$	$\times$	$\times$
$\lambda_f = \lambda_+$	$\circ$	
$\lambda_p < \lambda_f < \lambda_+$	$\circ$	$\circ$
$\lambda_f \leq \lambda_p$	$\circ$	$\times$

TABLE II: For $-\frac{1}{4}\xi a_g < \lambda_g < 2\xi a_g$

	inner	outer
$\lambda_+ < \lambda_f$	$\times$	$\times$
$\lambda_f = \lambda_+$	$\circ$	
$\lambda_p < \lambda_f < \lambda_+$	$\circ$	$\circ$
$\lambda_m < \lambda_f \leq \lambda_p$	$\circ$	$\times$
$\lambda_f \leq \lambda_m$	$\times$	$\times$

In the tables, "$\circ$" means there exists a positive root of $g(\epsilon) = 0$ which satisfies $H_0^2 > 0$, i.e., a de Sitter solution exists. And, "$\times$" means there exists no positive root or there exists a positive root for $g(\epsilon) = 0$ but $H_0^2 \leq 0$, i.e., no de Sitter solution exists. For $\lambda_g \leq -\frac{1}{4}\xi a_g$, there is no root satisfying $H_0^2 > 0$. Surprisingly, we have an upper bound for λ_f and there exist de Sitter solutions even for arbitrary large negative λ_f in the case $\lambda_g \geq 2\xi a_g$. We note that $\lambda_g - 2\xi a_g$ can be interpreted as an effective cosmological constant if we see the explicit constant term in the first line of (15). It is remarkable that there also exists a de Sitter solution for the case effective cosmological constant is zero.

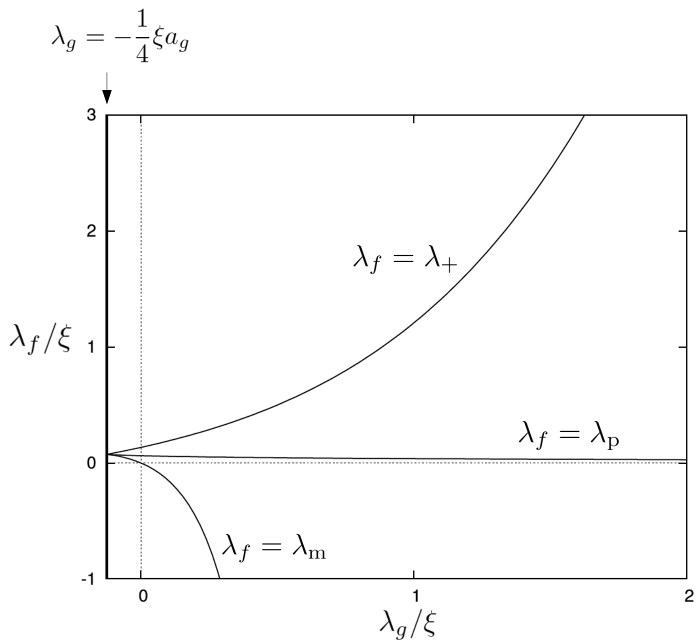

FIG. 3: We depicted the region de Sitter solutions exist. We chose $a_g = 0.5$. The multiple solution exists on $\lambda_f = \lambda_+$ curve. The outer root exists in the region below λ_+ and above λ_p. The inner root exists in the region below λ_+ and above λ_m. $\lambda_p \to +0$ as $\lambda_g \to +\infty$ and $\lambda_m \to -\infty$ as $\lambda_g \to 2\xi a_g - 0$ as we mentioned in the text. The triple point is given by $(\lambda_g,\ \lambda_f) = (-\frac{1}{4}\xi a_g,\ \frac{4}{27}\xi(1-a_g))$ and there $H_0^2 = 0$.

B. Stability of de Sitter solutions

In this subsection, we examine the stability of de Sitter solutions. In Sec. II B, We saw that the sign of $m_{\rm eff}^2$ determines the stability of de Sitter solutions, i.e., solutions are stable if $m_{\rm eff}^2$ is positive and unstable if $m_{\rm eff}^2$ is negative. Recalling the formula

$$m_{\rm eff}^2 = \xi\Big[a_g\epsilon_0 + (1-a_g)\frac{1}{\epsilon_0}\Big](3-2\epsilon_0)\ ,$$

we can see that $m_{\rm eff}^2$ is positive when $\epsilon_0 < \frac{3}{2}$ and $m_{\rm eff}^2$ is negative when $\epsilon_0 > \frac{3}{2}$. From now on, we suppose $\lambda_g > -\frac{1}{4}\xi a_g$ so that $H_0^2 > 0$ is satisfied.

We know that $g(\epsilon) = 0$ has positive roots when $\lambda_f \leq \lambda_+$. We first consider $\lambda_f = \lambda_+$ case where there exists a multiple positive root ϵ_*. Since we supposed $\lambda_g > -\frac{1}{4}\xi a_g$, we can evaluate ϵ_* as

$$\frac{3}{2} - \epsilon_* = \frac{3(\frac{\lambda_g}{\xi} + \frac{a_g}{4})}{(\frac{\lambda_g}{\xi} + \frac{a_g}{4}) + \frac{9}{4}a_g + (1-a_g) + \sqrt{\bar{c}^2 + 9a_g(1-a_g)}} > 0\ . \tag{57}$$

Then, $m_{\rm eff}^2$ is positive. Therefore, we find that the de Sitter solution corresponding to the multiple root is stable.

Next, we decrease λ_f from λ_+. The inner root always satisfies $\epsilon_{\rm in} < \epsilon_*$ as we mentioned in subsection IV A 2. We know ϵ_* is smaller than $\frac{3}{2}$. Therefore the inner root is always stable since $\epsilon_{\rm in} < \epsilon_* < \frac{3}{2}$. On the other hand, the outer root always satisfies $\epsilon_{\rm out} > \epsilon_*$. Since ϵ_* is smaller than $\frac{3}{2}$ and the outer root monotonically increase as λ_f decreases, we can expect that $\epsilon_{\rm out}$ sometime reaches $\frac{3}{2}$. Once $\epsilon_{\rm out}$ reaches $\frac{3}{2}$, the effective mass vanishes. There, λ_f is given by

$$\lambda_{\frac{3}{2}} = \frac{4}{27}\Big[3\Big(\lambda_g + \frac{\xi a_g}{4}\Big) + \xi(1-a_g)\Big] > 0 \ , \tag{58}$$

and Hubble scale reads

$$H_0^2\Big(\frac{3}{2}\Big) = \lambda_g + \frac{\xi a_g}{4} > 0 \ . \tag{59}$$

Note that $\lambda_p < \lambda_{\frac{3}{2}} < \lambda_+$ because $\epsilon_* < \frac{3}{2} < \epsilon_p$ (see (42) and (57)). Therefore, the outer root is stable when $\lambda_f \geq \lambda_{\frac{3}{2}}$ and unstable when $\lambda_f < \lambda_{\frac{3}{2}}$.

C. Appearance of the Higuchi bound

In this subsection, we will evaluate the effective mass of the massive graviton corresponding to the anisotropy.

From the definition of $m_{\rm eff}^2$ and the first line of (15), we can deduce the following expression

$$\begin{aligned} m_{\rm eff}^2(\epsilon_0) - 2H_0^2(\epsilon_0) &= -\frac{3\xi}{\epsilon_0}\left[a_g\epsilon_0^2 + \frac{2}{3}\bar{c}\epsilon_0 - (1-a_g)\right] \\ &= \frac{3\xi a_g}{\epsilon_0}(\epsilon_* - \epsilon_0)(\epsilon_0 - \epsilon_2) \ , \end{aligned} \tag{60}$$

where ϵ_* is given in (45) and we defined

$$\epsilon_2 = \frac{-\bar{c} - \sqrt{\bar{c}^2 + 9a_g(1-a_g)}}{3a_g} < 0 \ .$$

Since ϵ_2 is negative, the sign of $m_{\rm eff}^2 - 2H_0^2$ depends on that of $(\epsilon_* - \epsilon_0)$. Namely, $\epsilon_0 = \epsilon_*$ is equivalent to $m_{\rm eff}^2 = 2H_0^2$, $\epsilon_0 < \epsilon_*$ leads to $m_{\rm eff}^2 > 2H_0^2$, and $\epsilon_0 > \epsilon_*$ leads to $m_{\rm eff}^2 < 2H_0^2$. When $\lambda_f = \lambda_+$, the multiple root ϵ_* obviously satisfies $m_{\rm eff}^2 = 2H_0^2$. When $\lambda_f < \lambda_+$, there are two positive roots for $g(\epsilon) = 0$. The inner root always satisfies $\epsilon_{\rm in} < \epsilon_*$ as we mentioned in subsection IV A 2. Hence, the inner root always satisfies $m_{\rm eff}^2 > 2H_0^2$. On the other

hand, the outer root always satisfies $\epsilon_{\rm out} > \epsilon_*$. Therefore, the outer root always satisfies $m_{\rm eff}^2 < 2H_0^2$.

Remarkably, the equation $m_{\rm eff}^2 - 2H_0^2 = 0$ coincides with the equation determining the multiple root ϵ_* (see (44) and (60)). That is the reason why the bifurcation point of de Sitter solutions is exactly the same as the Higuchi bound.

Note that the anisotropy decays more rapidly than Hubble time scale $1/H_0$ for the inner root and it decays more slowly than $1/H_0$ or exponentially grows for the outer root if we use the analysis of Sec. II B.

Finally, we shall see that the ratio of the effective mass to Hubble scale monotonically varies along the line that the value of ϵ_0 is constant on λ_g-λ_f plane. We define ζ as the ratio of the effective mass to Hubble scale,

$$\zeta = \frac{m_{\rm eff}^2}{H_0^2} = \frac{\xi[a_g\epsilon_0 + (1-a_g)\frac{1}{\epsilon_0}](3-2\epsilon_0)}{\lambda_g + \xi a_g(2-\epsilon_0)(\epsilon_0 - 1)} \ . \tag{61}$$

From this expression, it is obvious that $\partial\zeta/\partial\lambda_g|_{\epsilon_0=\rm const.} < 0$ for $\epsilon_0 > \frac{3}{2}$ where ζ is positive, and $\partial\zeta/\partial\lambda_g|_{\epsilon_0=\rm const.} > 0$ for $\epsilon_0 < \frac{3}{2}$ where ζ is negative.

We will check how the line that ϵ_0 is constant can be drawn on λ_g-λ_f plane. When we fix the value of ϵ_0, $g(\epsilon_0) = 0$ gives the relation between λ_g and λ_f as

$$\lambda_f = \frac{1}{\epsilon_0^2}\lambda_g - \xi a_g + \frac{3a_g\xi\epsilon_0^2 + (1-3a_g)\xi\epsilon_0 - \xi(1-a_g)}{\epsilon_0^3} \ . \tag{62}$$

On λ_g-λ_f plane, each point in the region $\lambda_f < \lambda_+$ determines two lines: one for the inner root and the other for the outer root. From the fact that

$$\frac{{\rm d}\lambda_+}{{\rm d}\lambda_g} = \left(\frac{\bar{c} + \sqrt{\bar{c}^2 + 9a_g(1-a_g)}}{3(1-a_g)}\right)^2 = \frac{1}{\epsilon_*^2} \ , \tag{63}$$

each line is tangential to $\lambda_f = \lambda_+$ curve. We also know that $\lambda_f = \lambda_+$ is a convex function since from

$$\frac{{\rm d}\epsilon_*}{{\rm d}\lambda_g} = \frac{-\epsilon_*}{\xi\sqrt{\bar{c}^2 + 9a_g(1-a_g)}} < 0 \ , \tag{64}$$

we can obtain

$$\frac{{\rm d}^2\lambda_+}{{\rm d}\lambda_g^2} = \frac{-2}{\epsilon_*^3}\frac{{\rm d}\epsilon_*}{{\rm d}\lambda_g} > 0 \ . \tag{65}$$

Therefore, the series of lines cover the whole region satisfying $\lambda_f < \lambda_+$.

Using these formulae, we depicted Fig. 4 and Fig. 5

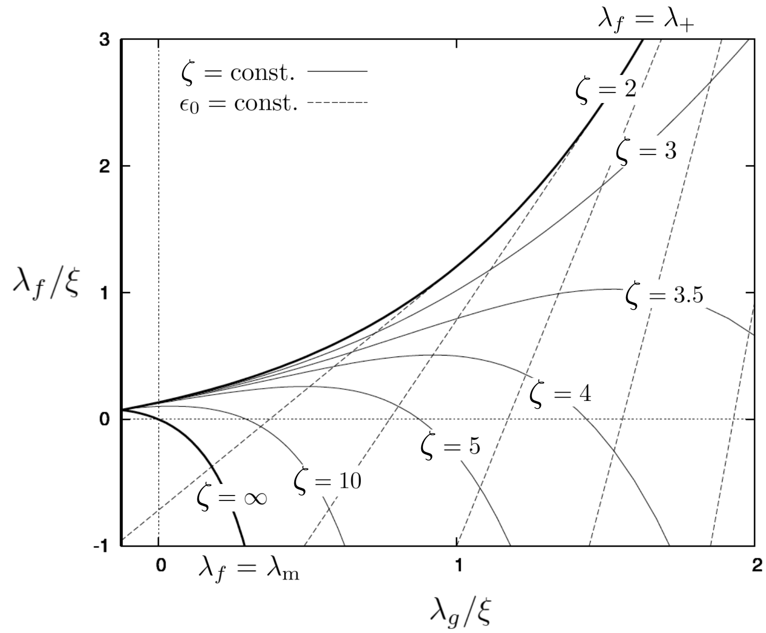

FIG. 4: We plotted ζ =const. curves of the inner root on λ_g-λ_f plane. We set $a_g = 0.5$. The inner root is always stable since $m_{\rm eff}^2$ is positive. In this figure, $\zeta = 2$ and $\zeta = +\infty$ curves coincide with $\lambda_f = \lambda_+$ and $\lambda_f = \lambda_m$ curves, respectively. Note that if we start from a point on $\lambda_f = \lambda_+$, ζ monotonically increases along $\epsilon_{\rm in} =$ const. line.

V. CONCLUSION

We investigated the cosmic no-hair conjecture in the presence of spin-2 matter. More precisely, we studied the cosmic no-hair conjecture in bimetric gravity using the perturbative method. First, we analyzed de Sitter solutions and found that there are two branches of de Sitter solutions. We examined the stability of de Sitter solutions and found that there always at least one stable solution. Finally, we evaluated the effective mass and found that the stable branch of de Sitter solutions satisfies the Higuchi bound. The other branch does not satisfy the Higuchi bound. The bifurcation point of two branches exactly coincides with the Higuchi bound. Thus, we concluded that there exists a de Sitter solution for which the anisotropy decays and the effective mass for these perturbations satisfy the Higuchi bound. Since the cosmic no-hair conjecture is already violated at the linear level in known cases, our result indicates that the cosmic no-hair conjecture is correct in bimetric gravity even though we have not given the nonlinear analysis.

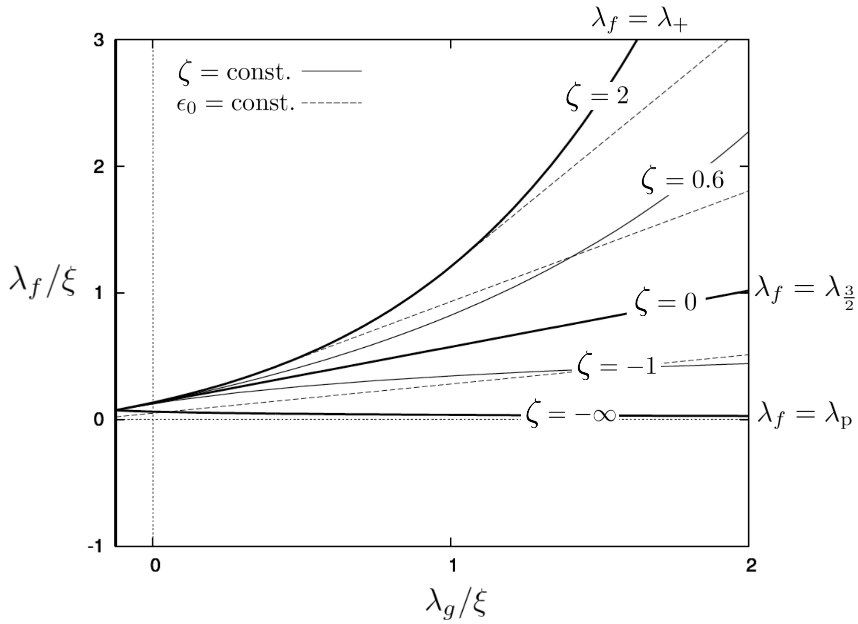

FIG. 5: We plotted ζ =const. curves of the outer root on λ_g-λ_f plane. We set $a_g = 0.5$. The outer root is stable above $\lambda_{\frac{3}{2}}$ where $m^2_{\rm eff}$ is positive and unstable below $\lambda_{\frac{3}{2}}$ where $m^2_{\rm eff}$ is negative. In this figure, $\zeta = 2$, $\zeta = 0$ and $\zeta = -\infty$ curves coincide with $\lambda_f = \lambda_+$, $\lambda_f = \lambda_{\frac{3}{2}}$ and $\lambda_f = \lambda_p$ curves, respectively. We see that if we start from a point on $\lambda_f = \lambda_+$, ζ monotonically decreases along $\epsilon_{\rm out} =$ const. line in the stable region and monotonically increases in the unstable region.

As a future work, it would be interesting to explore the meaning behind the curious fact that the bifurcation point of two branches of de Sitter solutions coincides with the Higuchi bound. Moreover, since we have found that at least one branch of de Sitter solutions is stable in our analysis, we can consider inflation in bimetric gravity without pathologies. It would be important to clarify what kind of signatures peculiar to bimetric gravity appear for example in the cosmic microwave background radiation.

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Appendix A: Derivation of Basic equations

In this appendix, we derive basic equations.

1. Ansatz and Lagrangian

We start with the anisotropic metric ansatz for $g_{\mu\nu}$ and $f_{\mu\nu}$

$$ds^2 = -N^2(t)dt^2 + e^{2\alpha(t)}[e^{-4\sigma(t)}dx^2 + e^{2\sigma(t)}(dy^2 + dz^2)] \ ,$$

and

$$ds'^2 = -M^2(t)dt^2 + e^{2\beta(t)}[e^{-4\lambda(t)}dx^2 + e^{2\lambda(t)}(dy^2 + dz^2)] \ ,$$

respectively. From these metrics, scalar curvatures are calculated as

$$R[g_{\mu\nu}] = \frac{1}{N^2}(-6\dot{\alpha}^2 + 6\dot{\sigma}^2) \ , \qquad R[f_{\mu\nu}] = \frac{1}{M^2}(-6\dot{\beta}^2 + 6\dot{\lambda}^2) \ . \tag{A1}$$

Moreover, $g^{-1}f$ is given by

$$g^{-1}f = \begin{pmatrix} (M/N)^2 & & & \\ & e^{2\beta-2\alpha-4\lambda+4\sigma} & & \\ & & e^{2\beta-2\alpha+2\lambda-2\sigma} & \\ & & & e^{2\beta-2\alpha+2\lambda-2\sigma} \end{pmatrix} = \begin{pmatrix} \gamma^2 & & & \\ & A^2 & & \\ & & B^2 & \\ & & & B^2 \end{pmatrix} ,$$

where we have defined variables as

$$\gamma = M/N \ , \qquad \epsilon = e^{\beta-\alpha} \ , \qquad \eta = e^{\lambda-\sigma} \ ,$$

$$A = \epsilon\eta^{-2} = e^{\beta-\alpha-2\lambda+2\sigma} \ , \qquad B = \epsilon\eta = e^{\beta-\alpha+\lambda-\sigma} \ .$$

Thus, we obtain

$$L = 1 - \sqrt{g^{-1}f} = \begin{pmatrix} 1-\gamma & & & \\ & 1-A & & \\ & & 1-B & \\ & & & 1-B \end{pmatrix} , \tag{A2}$$

Then, we can calculate the interaction term as

$$\begin{aligned} F_2 &= \frac{1}{2}\big([L]^2 - [L^2]\big) \\ &= \frac{1}{2}[(4-A-2B-\gamma)^2 - (4-2A-4B+A^2+2B^2-2\gamma+\gamma^2)] \\ &= [6-3A-6B+B(2A+B)+\gamma(-3+A+2B)] \ . \end{aligned} \tag{A3}$$

Therefore, the Lagrangian reads

$$\begin{aligned} \mathcal{L} = \ & M_g^2 e^{3\alpha}\Big[\frac{3}{N}(-\dot{\alpha}^2+\dot{\sigma}^2) - N\Lambda_g\Big] + M_f^2 e^{3\beta}\Big[\frac{3}{M}(-\dot{\beta}^2+\dot{\lambda}^2) - M\Lambda_f\Big] \\ & + m^2 M_e^2 N e^{3\alpha}\big[6-3A-6B+B(2A+B)+\gamma(-3+A+2B)\big] \ . \end{aligned} \tag{A4}$$

2. Equations of motion and constraints

We normalize parameters and time with M_e as follows:

$$a_g = \frac{M_e^2}{M_g^2} \ , \qquad \xi = \frac{m^2}{M_e^2} \ , \qquad \lambda_g = \frac{\Lambda_g}{3M_e^2} \ , \qquad \lambda_f = \frac{\Lambda_f}{3M_e^2} \ , \qquad {}' = \cdot/M_e \ .$$

Note that $0 < a_g < 1$ from the definition of M_e.

From the Lagrangian, we obtain the equations of motion

$$\left(\frac{\alpha'}{N}\right)' + 3\frac{\sigma'^2}{N} + \frac{1}{6}\xi a_g[N(3A+6B-2B(2A+B)) - M(9-2A-4B)] = 0 \ , \tag{A5}$$

$$\left(\frac{\beta'}{M}\right)' + 3\frac{\lambda'^2}{M} - \frac{1}{6}\xi(1-a_g)\frac{1}{\epsilon^3}[N(3A+6B-2B(2A+B)) - M(9-2A-4B)] = 0 \ , \tag{A6}$$

$$\left(\frac{\sigma'}{N}\right)' + 3\frac{\alpha'\sigma'}{N} + \frac{1}{3}\xi a_g(A-B)[N(3-B)-M] = 0 \ , \tag{A7}$$

$$\left(\frac{\lambda'}{M}\right)' + 3\frac{\beta'\lambda'}{M} - \frac{1}{3}\xi(1-a_g)\frac{1}{\epsilon^3}(A-B)[N(3-B)-M] = 0 \ , \tag{A8}$$

and the constraints

$$\left(\frac{\alpha'}{N}\right)^2 - \left(\frac{\sigma'}{N}\right)^2 = \lambda_g + \frac{1}{3}\xi a_g[-6+3A+6B-B(2A+B)] \ , \tag{A9}$$

$$\left(\frac{\beta'}{M}\right)^2 - \left(\frac{\lambda'}{M}\right)^2 = \lambda_f + \frac{1}{3}\xi(1-a_g)\frac{1}{\epsilon^3}(3 - A - 2B) \ . \tag{A10}$$

It is easy to find the consistency relation

$$\frac{M}{N} = \frac{\beta'(3A + 6B - 2B(2A + B)) - \lambda'(2A - 2B)(3 - B)}{\alpha'(9 - 2A - 4B) - \sigma'(2A - 2B)} \ . \tag{A11}$$

From the linear combination of (A7) and (A8), we can also obtain a conserved quantity

$$E = \frac{1}{a_g}\frac{e^{3\alpha}\sigma'}{N} + \frac{1}{1-a_g}\frac{e^{3\beta}\lambda'}{M} \ . \tag{A12}$$

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[1] R. M. Wald, Phys. Rev. D **28**, 2118 (1983).

[2] M. -a. Watanabe, S. Kanno and J. Soda, Phys. Rev. Lett. **102**, 191302 (2009) [arXiv:0902.2833 [hep-th]].

[3] M. -a. Watanabe, S. Kanno and J. Soda, Prog. Theor. Phys. **123**, 1041 (2010) [arXiv:1003.0056 [astro-ph.CO]].

[4] C. J. Isham, A. Salam and J. A. Strathdee, Phys. Rev. D **3**, 867 (1971).

[5] C. de Rham and G. Gabadadze, Phys. Rev. D **82**, 044020 (2010) [arXiv:1007.0443 [hep-th]].

[6] C. de Rham, G. Gabadadze and A. J. Tolley, Phys. Rev. Lett. **106**, 231101 (2011) [arXiv:1011.1232 [hep-th]].

[7] S. F. Hassan and R. A. Rosen, Phys. Rev. Lett. **108**, 041101 (2012) [arXiv:1106.3344 [hep-th]].

[8] S. F. Hassan and R. A. Rosen, JHEP **1202**, 126 (2012) [arXiv:1109.3515 [hep-th]].

[9] S. F. Hassan and R. A. Rosen, JHEP **1204**, 123 (2012) [arXiv:1111.2070 [hep-th]].

[10] M. S. Volkov, JHEP **1201**, 035 (2012) [arXiv:1110.6153 [hep-th]].

[11] A. Higuchi, Nucl. Phys. B **282**, 397 (1987).

[12] C. de Rham and S. Renaux-Petel, arXiv:1206.3482 [hep-th].

[13] M. Fasiello and A. J. Tolley, arXiv:1206.3852 [hep-th].

[14] S. F. Hassan, A. Schmidt-May and M. von Strauss, arXiv:1208.1797 [hep-th].

[15] D. G. Boulware and S. Deser, Phys. Rev. D **6**, 3368 (1972).

[16] T. Damour, I. I. Kogan and A. Papazoglou, Phys. Rev. D **66**, 104025 (2002) [hep-th/0206044].

[17] T. Damour and I. I. Kogan, Phys. Rev. D **66**, 104024 (2002) [hep-th/0206042].

[18] N. Khosravi, N. Rahmanpour, H. R. Sepangi and S. Shahidi, Phys. Rev. D **85**, 024049 (2012) [arXiv:1111.5346 [hep-th]].

[19] K. Nomura and J. Soda, arXiv:1207.3637 [hep-th].

[20] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A **173**, 211 (1939).

[21] M. von Strauss, A. Schmidt-May, J. Enander, E. Mortsell and S. F. Hassan, JCAP **1203**, 042 (2012) [arXiv:1111.1655 [gr-qc]].

[22] D. Comelli, M. Crisostomi, F. Nesti and L. Pilo, JHEP **1203**, 067 (2012) [Erratum-ibid. **1206**, 020 (2012)] [arXiv:1111.1983 [hep-th]].

[23] M. S. Volkov, Phys. Rev. D **86**, 061502 (2012) [arXiv:1205.5713 [hep-th]].

[24] M. Berg, I. Buchberger, J. Enander, E. Mortsell and S. Sjors, arXiv:1206.3496 [gr-qc].

[25] Y. Akrami, T. S. Koivisto and M. Sandstad, arXiv:1209.0457 [astro-ph.CO].

[26] A. De Felice, A. E. Gumrukcuoglu and S. Mukohyama, Phys. Rev. Lett. **109**, 171101 (2012) [arXiv:1206.2080 [hep-th]].

[27] A. E. Gumrukcuoglu, C. Lin and S. Mukohyama, Phys. Lett. B **717**, 295 (2012) [arXiv:1206.2723 [hep-th]].

[28] G. Tasinato, K. Koyama and G. Niz, arXiv:1210.3627 [hep-th].

[29] A. E. Gumrukcuoglu, C. Lin and S. Mukohyama, JCAP **1203**, 006 (2012) [arXiv:1111.4107 [hep-th]].

[30] D. Comelli, M. Crisostomi and L. Pilo, JHEP **1206**, 085 (2012) [arXiv:1202.1986 [hep-th]].